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UNIT - 1

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Q1] Verify Rolle's Theorem for $f(x) = x^2$ in $[-1, 1]$

→ Let, $f(x) = x^2$

① $f(x) = x^2$ is continues fxn in $[-1, 1]$

② $f(x) = x^2$ is diff. in $(-1, 1)$

③ $f(1) = 1^2 = 1$

$f(-1) = (-1)^2 = 1$

$f(-1) = f(1) = 1$

$\exists$ at least one point $c \in (-1, 1)$

$f'(c) = 0$

$f'(c) = 2c, \quad c = 0, \quad 0 \in (-1, 1)$

Q2] Using LMVT, show that if $0 < a < b$

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$$

→ Let, $f(x) = \sin^{-1}(x), \quad x \in (a, b)$

① It is continues in $[a, b]$

② It is differential in (a, b)

$$f'(x) = \frac{1}{\sqrt{1-x^2}}, \quad a < x < b$$

③ $\exists$ at least one point, $c \in (a, b)$

$$f'(c) = \frac{f(b) - f(a)}{b-a} \quad \text{---} ①$$

$$f'(c) = \frac{1}{\sqrt{1-c^2}}$$

$f(a) = \sin^{-1}(a)$

$f(b) = \sin^{-1}(b)$

--- (put all value in ①)

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}b - \sin^{-1}a}{b-a}$$

$a < c < b$ --- squaring

$a^2 < c^2 < b^2$ --- Multiply by (-) sign

$-a^2 < -c^2 < -b^2$ --- add 1

$\sqrt{1-a^2} < \sqrt{1-c^2} < \sqrt{1-b^2}$ --- Take underroot

$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$ --- Take reciprocal

--- (put value of $\frac{1}{\sqrt{1-c^2}}$)

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}b - \sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

--- (Multiply by $(b-a)$)

$$\boxed{\frac{(b-a)}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{(b-a)}{\sqrt{1-b^2}}}$$

Hence proved.

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q 3] Expand $(1+x)^{1/x}$ upto power of x^2.

$\Rightarrow$ $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$y = (1+x)^{1/x}$ .... (Taking log on BS)

$\log y = \frac{1}{x}\log(1+x)$ -- (substitute the series of $\log(1+x)$)

$\log y = \frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \right]$

$\log y = \left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots\right]$

$\log y = z$

Here, $z = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots$

$\rightarrow y = e^z$

.... $\left[e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots\right]$

$y = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} \dots$

$y = 1 + \left[1 - \frac{x}{2} + \frac{x^2}{3} \dots\right] + \frac{1}{2!}\left[1 - \frac{x}{2} + \frac{x^2}{3} \dots\right]^2$

$+ \frac{1}{3!}\left[1 - \frac{x}{2} + \frac{x^2}{3} - \dots\right]^3$

$= 1 + \left[1 - \frac{x}{2} + \frac{x^2}{3} \dots\right] + \frac{1}{2!}\left[1 + \frac{x^2}{4} - x\right] + \frac{1}{3!}\left[1 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{x^3}{4}\right]$

$= 2 - \frac{x}{2} + \frac{x^2}{2} + \frac{1}{2} + \frac{x^2}{2} - \frac{x}{2} + \frac{1}{6} - \frac{x}{4} + \frac{x^2}{8}$

$= \frac{8}{3} - \frac{5x}{4} + \frac{11x^2}{24} - \dots$

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Q4] Expand $\log(1+\tan x)$ up to term x^3

→ Let, $f(x) = \log(1+\tan x)$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$

$$\log(1+\tan x) = \tan x - \frac{\tan^2 x}{2} + \frac{\tan^3 x}{3} + \cdots$$

$$= \left(x+\frac{x^3}{3}\right) - \frac{1}{2}\left(x+\frac{x^3}{3}\right)^2 + \frac{1}{3}\left(x+\frac{x^3}{3}\right)^3$$

$$= \left(x+\frac{x^3}{3}\right) - \frac{1}{2}\left(x^2 + \frac{x^6}{9} + \frac{2x^4}{3}\right) + \frac{1}{3}\left(x^3 + \frac{x^9}{27} + \frac{3x^2x^3}{3} + \frac{3 \times x^6}{9}\right)$$

$$= x + \frac{x^3}{3} - \frac{x^2}{2} - \frac{x^6}{18} + \frac{x^3}{3} + \frac{x^5}{3}$$

$$= x - \frac{x^2}{2} + \frac{2x^3}{3}$$

Q5] By using Taylor's theorem, expand $f(x) = e^x$ in powers of $(x-2)$

→ By Taylor's series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$

where, $\boxed{a=2}$

① $f(x) = e^x$
$f(2) = e^2 = 7.38$

② $f'(x) = e^x$
$f'(2) = e^2 = 7.38$

③ $f''(x) = e^x$
$f''(2) = e^2 = 7.38$

④ $f'''(x) = e^x$
$f'''(2) = e^2 = 7.38$

⑤ $f^{iv}(x) = e^x$
$f^{iv}(2) = e^2 = 7.38$

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$$f(x) = 7.38 + (x-2)(7.38) + \frac{(x-2)^2}{2}(7.38) + \frac{(x-2)^3}{6}(7.38) + \frac{(x-2)^4}{24}(7.38)$$

$$f(x) = 7.38 + (7.38)(x-2) + (3.69)(x-2)^2 + (1.23)(x-2)^3$$

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q6] Use Taylor's theorem $2x^3+7x^2+x-6$ in powers of $(x+2)$

→ Let, $f(x) = 2x^3+7x^2+x-6$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{2!}f'''(a)$$

where, $\boxed{a=-2}$

① $f(x) = 2x^3+7x^2+x-6$
$f(-2) = 2(-2)^3 + 7(-2)^2 - 2 - 6 = 4$

② $f'(x) = 6x^2+14x+1$
$f'(-2) = 6(-2)^2 + 14(-2) + 1 = -3$

③ $f''(x) = 12x + 14$
$f''(-2) = 12(-2) + 14 = -10$

④ $f'''(x) = 12$
$f'''(-2) = 12$

∴ (put all values in series)

$$f(x) = f(-2) + (x+2)f'(-2) + \frac{(x+2)^2}{2!}f''(-2) + \frac{(x+2)^3}{3!}f'''(-2)$$

$$= 4 + (x+2)(-3) + \frac{(x+2)^2}{2}(-10) + \frac{(x+2)^3}{6}(12)$$

$$\boxed{f(x) = 2(x+2)^3 - 5(x+2)^2 - 3(x+2) + 4}$$

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Q.1] Using Taylors theorem expand $2x^3 + 3x^2 - 8x + 7$ in power of $(x-2)$

→ Let, $f(x) = 2x^3 + 3x^2 - 8x + 7$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

By taylor series, on comparision we get

$\boxed{a = 2}$

① $f(x) = 2x^3 + 3x^2 - 8x + 7$

$f(2) = 2(2)^3 + 2(2)^2 - 8(2) + 7 = 15$

② $f'(x) = 6x^2 + 6x - 8$

$f'(2) = 6(2)^2 + 6(2) - 8 = 28$

③ $f''(x) = 12x + 6$

$f''(2) = 12(2) + 6 = 30$

④ $f'''(x) = 12$

$f'''(2) = 12$

⑤ $f^{IV}(x) = 0$

$f^{IV}(2) = 0$ ... (put all value in series)

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2)$$

$$= 15 + (x-2)(28) + \frac{(x-2)^2}{2}(30) + \frac{(x-2)^3}{6}(12) + \dots$$

$\boxed{f(x) = 2(x-2)^3 + 15(x-2)^2 + 28(x-2) + 15}$

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q.8] Find the Fourier series to represent the function $f(x) = x^2$ in the interval $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$ for all x

$\rightarrow$

Let, $f(x) = x^2$

$\therefore f(-x) = (-x)^2$

$\therefore f(-x) = x^2$

$\therefore f(-x) = f(x)$

$\therefore$ It is an even function

$\therefore b_n = 0$

$\therefore a_0, a_n = ?$

$f(x+2\pi) = f(x)$

$2\pi = 2L$

$\boxed{L = \pi}$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

① $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$= \frac{2}{\pi}\int_0^{\pi} (x^2)\,dx$

$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$

$= \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right]$

$\boxed{a_0 = \frac{2\pi^2}{3}}$

② $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} (x^2)\cos nx\,dx$

$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$

$$\frac{2}{\pi}\left[\pi^2\left(\frac{\sin n\pi}{n}\right)+2\pi\left(\frac{\cos n\pi}{n^2}\right)-2\left(\frac{\sin n\pi}{n^3}\right)\right]$$

$$-\left[0^2\,\frac{\sin n(0)}{n}+2(0)\,\frac{\cos n(0)}{n^2}-2\,\frac{\sin n(0)}{n^3}\right]$$

...[we know, :- $\cos n\pi = (-1)^n$, $\sin n\pi = 0$]

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}(\cos n\pi)\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}(-1)^n\right]$$

$$a_n = \frac{4(-1)^n}{n^2}$$

...(put value of a_0 & a_n in fourier series)

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)\cos nx$$

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Q.9] Find the Fourier series to represent the function $f(x) = x$ in the interval $-\pi < x < \pi$

$\Rightarrow$

Let, $f(x) = x$ where, $\boxed{L = \pi}$

$\therefore f(-x) = -x$

$\therefore f(-x) = -f(x)$

$\therefore$ It is an odd function

$\therefore a_0$ & $a_n = 0$

$\therefore b_n = ?$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

① $$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cdot \sin nx \, dx$$

$$= \frac{2}{\pi} \left[x \frac{(-\cos nx)}{n} - 1 \frac{(-\sin nx)}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{-\pi \cos n\pi}{n} + \frac{\sin n\pi}{n^2} \right) - \left(\frac{-(0)\cos n(0)}{n} + \frac{\sin n(0)}{n^2} \right) \right]$$

$\sin n\pi$ and $\sin n(0)$ are 0.

$\cdots \left[\cos n\pi = (-1)^n, \quad \sin n\pi = 0, \sin 0 = 0 \right]$

$$= \frac{2}{\pi} \left[-\pi \left(\frac{\cos n\pi}{n} \right) \right]$$

$$= \frac{2}{\pi} \left(\frac{-\pi (-1)^n}{n} \right)$$

$$\boxed{b_n = \frac{-2(-1)^n}{n}} \quad \cdots \text{(put in series)}$$

$$\boxed{f(x) = x = \sum_{n=1}^{\infty} \left(\frac{-2(-1)^n}{n} \right) (\sin nx)}$$

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q10] Find Half range sine series for the function $f(x) = x^2, \quad 0 < x < 2$

→ Let, $f(x) = x^2, \quad 0 < x < 2$

Half range sine :- $a_0, a_n = 0$, $\boxed{L = 2}$

$b_n = ?$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

① $$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{2}{2} \int_0^2 (x^2) \sin \frac{n\pi x}{2} \, dx$$

$$= \left[x^2 \left(-\cos \frac{n\pi x}{2}\right) \times \frac{1}{n\pi/2} + 2x \left(-\sin \frac{n\pi x}{2}\right) \times \frac{1}{n^2\pi^2/4} - 2 \left(\cos \frac{n\pi x}{2}\right) \times \frac{1}{n^3\pi^3/8} \right]_0^2$$

$$= \left[\frac{-2x^2}{n\pi} \left(\cos \frac{n\pi x}{2}\right) - \frac{8x}{n^2\pi^2} \left(\sin \frac{n\pi x}{2}\right) - \frac{16}{n^3\pi^3} \left(\cos \frac{n\pi x}{2}\right) \right]_0^2$$

--- (put limits)

$$= \left[\frac{-2(2)^2}{n\pi} \left(\cos \frac{n\pi(2)}{2}\right) - \frac{8(2)}{n^2\pi^2} \left(\sin \frac{n\pi(2)}{2}\right)^{0} - \frac{16}{n^3\pi^3} \left(\cos \frac{n\pi(2)}{2}\right) \right]$$

$$- \left[\frac{-2(0)}{n\pi} \left(\cos \frac{n\pi(0)}{2}\right)^{0} - \frac{8(0)}{n^2\pi^2} \left(\sin \frac{n\pi(0)}{2}\right)^{0} - \frac{16}{n^3\pi^3} \left(\cos \frac{n\pi(0)}{2}\right) \right]$$

--- $\left[\cos n\pi = (-1)^n, \ \cos 0 = 1 \right]$
$\left[\sin n\pi = 0, \ \sin 0 = 0 \right]$

$$= \left[\frac{-8}{n\pi} (\cos n\pi) - \frac{16}{n^3\pi^3} (\cos n\pi) + \frac{16}{n^3\pi^3} (\cos(0)) \right]$$

$$b_n = \frac{-8(-1)^n}{n\pi} - \frac{16(-1)^n}{n^3\pi^3} + \frac{16}{n^3\pi^3}$$

--- (put in series)

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{-8(-1)^n}{n\pi} \; \frac{-16(-1)^n}{n^3\pi^3} \; \frac{+16}{n^3\pi^3} \right) \sin \frac{n\pi x}{2}$$

Q.11] Find Half range cosine series for $f(x) = (x-1)^2$ in the interval $(0,1)$

$\Rightarrow$ Let, $f(x) = (x-1)^2$ where, $L = 1$

$\therefore$ It is an even function

$\therefore b_n = 0$

$\therefore a_0$ & $a_n = ?$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} \right]$$

① $$a_0 = \frac{2}{L} \int_0^L f(x)\, dx$$

$$= \frac{2}{1} \int_0^1 (x-1)^2 \, dx$$

$$= 2 \int_0^1 (x^2 + 1 - 2x)\, dx$$

$$= 2 \left[\int_0^1 x^2\, dx + \int_0^1 dx - 2\int_0^1 x\, dx \right]$$

$$= 2 \left[\frac{x^3}{3} + x - \frac{2x^2}{2} \right]_0^1$$

$$= 2 \left[\frac{1}{3} + 1 - \frac{2(1)^2}{2} \right] - [0 + 0 - 0]$$

$$= 2 \left[\frac{4}{3} - 1 \right]$$

$$= 2 \left[\frac{1}{3} \right]$$

$$\boxed{a_0 = \frac{2}{3}}$$

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(2) $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

$= \frac{2}{1}\int_0^1 (x-1)^2\cos\frac{n\pi x}{1}\,dx$

$= 2\left[\frac{(x-1)^2}{n\pi}\left(\sin n\pi x\right) - \frac{2(x-1)(1)}{n^2\pi^2}\left(-\cos n\pi x\right) + \frac{2}{n^3\pi^3}\left(-\sin n\pi x\right)\right]_0^1$

$= 2\left[\frac{(1-1)^2}{n\pi}\left(\sin n\pi(1)\right) + \frac{2(1-1)}{n^2\pi^2}\left(\cos n\pi(1)\right) - \frac{2}{n^3\pi^3}\left(\sin n\pi(1)\right)\right.$

$\left. - \left[\frac{(0-1)^2}{n\pi}\sin n\pi(0) - \frac{2(0-1)(1)}{n^2\pi^2}\left(-\cos n\pi(0)\right) + \frac{2}{n^2\pi^3}\right.\right.$

$\left.\left. -\sin n\pi(0)\right]\right]$

$\cos n\pi = (-1)^n$ & $\sin n\pi = 0$

$= 2\left[+\frac{2}{n^2\pi^2}\cos(0)\right]$

$$\boxed{a_n = \frac{4}{n^2\pi^2}}$$

---(put a_0 & a_n in series)

$$\boxed{F(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\right)\left(\cos\frac{n\pi x}{L}\right)}$$

Unit -2

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q] Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for $u = \tan^{-1}\frac{y}{x}$

$\Rightarrow$ LHS :- $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

- $\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2} \times \frac{1}{x} = \left(\frac{x^2}{x^2+y^2}\right)\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$
- $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$

$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

RHS :- $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

- $\frac{\partial u}{\partial x} = \frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right) = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$
- $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$

$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$

$= \frac{-x^2+y^2}{(x^2+y^2)^2}$

$\Rightarrow$ $\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

$\therefore \frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

LHS = RHS

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Q.2] If $u = (x^2 - y^2) f(xy)$ then find $u_{xx} + u_{yy}$

① $u_{xx} = \frac{\partial^2 u}{\partial x^2}$

$\frac{\partial u}{\partial x} = (x^2 - y^2) \cdot f'(xy) \cdot y + (2x) f(xy)$

$\frac{\partial^2 u}{\partial x^2} = (x^2 y - y^3) f'(xy) + (2x) f(xy)$

$= (x^2 y - y^3) f''(xy)(y) + f'(xy)(2xy)$
$+ (2x) f'(xy)(y) + f(xy)(2)$

* $u_{xx} = (x^2 y^2 - y^4) f''(xy) + 4xy f'(xy) + 2 f(xy)$

② $u_{yy} = \frac{\partial^2 u}{\partial y^2}$

$\frac{\partial u}{\partial y} = (x^2 - y^2) f'(xy)(x) + (-2y) f(xy)$

$= (x^3 - xy^2) f'(xy) - 2y f(xy)$

$\frac{\partial^2 u}{\partial y^2} = (x^3 - xy^2) f''(xy)(x) + f''(xy)(-2xy)$
$- 2y f'(xy)(x) + f(xy)(-2)$

$= (x^4 - x^2 y^2) f''(xy) - 2xy f'(xy)$
$- 2xy f'(xy) - 2 f(xy)$

* $u_{yy} = (x^4 - x^2 y^2) f''(xy) - 4xy f'(xy) - 2 f(xy)$

$\Rightarrow u_{xx} + u_{yy} \Rightarrow (x^2 y^2 - y^4) f''(xy) + 4xy f'(xy) + 2 f(xy)$
$+ (x^4 - x^2 y^2) f''(xy) - 4xy f'(xy) - 2 f(xy)$
$= f''(xy)(x^2 y^2 - y^4 + x^4 - x^2 y^2)$
$= (x^4 - y^4) f''(xy)$

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Q4] If $u = \operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ then show that :-

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$$

→ Let, $u = \operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

$$\operatorname{cosec} u = \sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

put, $x = xt$

$y = yt$

$$\operatorname{cosec} u = \sqrt{\frac{(xt)^{1/2}+(yt)^{1/2}}{(xt)^{1/3}+(yt)^{1/3}}}$$

$$= \frac{t^{1/4}}{t^{1/6}}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

$$= t^{1/12}$$

∴ It is homogenus eqn of degree $\left(\frac{1}{12}\right)$

By Euler's Theorem :-

① $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}$$

∴ $f(u) = \operatorname{cosec} u$

$f'(u) = -\operatorname{cosec} u \cot u$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{12}\left[\frac{\operatorname{cosec} u}{-\operatorname{cosec} u \cot u}\right]$$

$$= \frac{-1}{12}\left[\frac{1}{\cot u}\right] = \frac{-1}{12}(\tan u)$$

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(c) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

$\therefore g(u) = -\frac{\tan u}{12}$

$\therefore g'(u) = -\frac{\sec^2 u}{12}$

$= -\frac{\tan u}{12}\left[-\frac{\sec^2 u}{12} - 1\right]$

$= -\frac{\tan u}{12}\left[\frac{-1}{12}[1 + \tan^2 u] - 1\right]$

$= -\frac{\tan u}{12}\left[\frac{-1}{12} - \frac{\tan^2 u}{12} - 1\right]$

$= -\frac{\tan u}{12}\left[\frac{-13}{12} - \frac{\tan^2 u}{12}\right]$

$= \frac{\tan u}{12}\left[\frac{13}{12} + \frac{\tan^2 u}{12}\right]$

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Q.6] If $x = u\tan v$, $y = u\sec v$ then prove that $(ux)_y (vx)_y = (uy)_x (vy)_x$

$\Rightarrow$

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$$

LHS :- $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y$

① we have, $x = u\tan v$, $y = u\sec v$ } squaring

$x^2 = u^2\tan^2 v$

$-y^2 = -u^2 \cdot \sec^2 v$ } subtract

$x^2 - y^2 = u^2(\tan^2 v - \sec^2 v)$

$$\boxed{x^2 - y^2 = u^2} \quad \text{---} (1)$$

② $x = u\tan v$, $y = u\sec v$ } Dividing

$\frac{x}{y} = \frac{\sin v}{\cos v} \times \cos v$

$\frac{x}{y} = \sin v$

$\left(\frac{\partial u}{\partial x}\right)_y = ?$

$2u\left(\frac{\partial u}{\partial x}\right)_y = -2x$ --- from (1)

$\left(\frac{\partial u}{\partial x}\right)_y = \frac{-x}{u} = -\frac{x}{y}$

$\left(\frac{\partial v}{\partial x}\right)_y = ?$

$\frac{1}{y} = \cos v \left(\frac{\partial y}{\partial x}\right)_y$

$\left(\frac{\partial v}{\partial x}\right)_y = \frac{1}{y \cos v} = \frac{\sec v}{y}$

LHS $= \left(\frac{\partial y}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y$

$= \left(\frac{-x}{y}\right)\left(\frac{\sec v}{y}\right)$

RHS $= \left(\frac{\partial y}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$

$\left(\frac{\partial y}{\partial y}\right)_x = ?$

$2y \left(\frac{\partial y}{\partial y}\right)_x = 2y$

$\left(\frac{\partial y}{\partial y}\right)_x = \frac{xy}{xy} = \frac{y}{y}$

$\left(\frac{\partial v}{\partial y}\right)_x = ?$

$\cos v \left(\frac{\partial v}{\partial y}\right)_x = \frac{-x}{y^2}$

$\left(\frac{\partial v}{\partial y}\right)_x = \frac{-x}{y^2}\left(\frac{1}{\cos v}\right) = \frac{-x \sec v}{y^2}$

RHS $= \left(\frac{\partial y}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y$

$= \left(\frac{y}{y}\right)\left(\frac{-x \sec v}{y^2}\right)$

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LHS = RHS

$$\left(\frac{-x}{u}\right)\left(\frac{\sec v}{y}\right) = \left(\frac{y}{u}\right)\left(\frac{-x\sec v}{y^2}\right)$$

$$\left(\frac{-x}{u}\right)\left(\frac{y}{u}\right)\left(\frac{1}{y}\right) = \left(\frac{y}{u}\right)\left(\frac{-x}{y^2}\right)\left(\frac{y}{u}\right) \quad \text{--} \left(\sec v = \frac{y}{u}\right)$$

$$\frac{-x}{y^2} = \frac{-x}{y^2}$$

Q.7] If $u = ax + by$, $v = bx - ay$ prove that.

$$\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{a^2+b^2}{a^2}$$

$\Rightarrow$

① $\left(\frac{\partial y}{\partial v}\right)_x = ?$

$v = bx - ay$

$ay = bx - v$

$y = \frac{bx}{a} - \frac{v}{a}$

$\left(\frac{\partial y}{\partial v}\right)_x = \frac{-1}{a}$

② $\left(\frac{\partial v}{\partial y}\right)_u = ?$

$(u = ax + by) \times b$

$(v = bx - ay) \times a$

$bu = abx + b^2y$

$av = abx - a^2y$

$- \quad - \quad +$

$bu - av = y(a^2+b^2)$

~~sujeet~~

$bu - av = y(a^2+b^2)$

$bu - y(a^2+b^2) = av$

$\frac{bu}{a} - (a^2+b^2)\frac{y}{a} = v$

$\left(\frac{\partial v}{\partial y}\right)_u = -\frac{(a^2+b^2)}{a}$

$\Rightarrow \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u$

$= \left(\frac{-1}{a}\right)\left(\frac{-(a^2+b^2)}{a}\right)$

$= \frac{a^2+b^2}{a}$

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Q.8] If $u = f(r,s)$, $r = x^2+y^2$, $s = x^2-y^2$ prove that, $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 4xy\frac{\partial u}{\partial r}$

$\Rightarrow$ $u \longrightarrow (r,s) \longrightarrow (x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} \quad \text{---(1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \quad \text{---(2)}$$

(1) $\frac{\partial r}{\partial x} = 2x$ | (2) $\frac{\partial s}{\partial x} = 2x$

(1) $\frac{\partial r}{\partial y} = 2y$ | (2) $\frac{\partial s}{\partial y} = -2y$

--- (put in (1) & (2))

$$\frac{\partial u}{\partial x} = (2x)\frac{\partial u}{\partial r} + (2x)\left(\frac{\partial u}{\partial s}\right) \quad \text{---(3)}$$

$$\frac{\partial u}{\partial y} = (2y)\frac{\partial u}{\partial r} - (2y)\left(\frac{\partial u}{\partial s}\right) \quad \text{---(4)}$$

--- (Multiply (3) by y & Multiply (4) by x) and add it)

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = (2xy)\frac{\partial u}{\partial r} + (2xy)\frac{\partial u}{\partial s} + 2xy\frac{\partial u}{\partial r} - 2xy\left(\frac{\partial u}{\partial s}\right)$$

$$\boxed{y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 4xy\frac{\partial u}{\partial r}}$$

Hence proved

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Q.5] If $u = x^2 - y^2$, $v = 2xy$, $z = f(u,v)$ then show that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2\sqrt{u^2+v^2}\,\frac{\partial z}{\partial u}$

→ $z \to u, v \to x, y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad ---(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \quad ---(2)$$

① $\frac{\partial u}{\partial x} = 2x$ ② $\frac{\partial v}{\partial x} = 2y$ } Put in (1)

③ $\frac{\partial u}{\partial y} = -2y$ ④ $\frac{\partial v}{\partial y} = 2x$ } put in (2)

We get,

$$\frac{\partial z}{\partial x} = (2x)\frac{\partial z}{\partial u} + (2y)\frac{\partial z}{\partial v} \quad ----(3)$$

$$\frac{\partial z}{\partial y} = (-2y)\frac{\partial z}{\partial u} + (2x)\frac{\partial z}{\partial v} \quad ---(4)$$

----(equ ③ × x and equ ④ × y and subtract it)

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x^2\frac{\partial z}{\partial u} + 2xy\frac{\partial z}{\partial v} + 2y^2\frac{\partial z}{\partial u} - 2xy\frac{\partial z}{\partial v}$$

$$= 2x^2\frac{\partial z}{\partial u} + 2y^2\frac{\partial z}{\partial u}$$

$$= 2(x^2+y^2)\frac{\partial z}{\partial u}$$

$(\sqrt{u^2+v^2}) = (x^2-y^2)^2 + (2xy)^2$

$u^2+v^2 = x^4+y^4-2x^2y^2+4x^2y^2 = (x^2+y^2)^2$

$$\boxed{x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2\sqrt{u^2+v^2}\,\frac{\partial z}{\partial u}}$$

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q10] If $u = \sin^{-1}\sqrt{\frac{x^2+y^2}{x+y}}$ show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}\tan u(\tan^2 u - 1)$$

$\Rightarrow$ Let, $u = \sin^{-1}\sqrt{\frac{x^2+y^2}{x+y}}$

$$\sin u = \sqrt{\frac{x^2+y^2}{x+y}} \qquad x = xt, \quad y = yt$$

$$= \sqrt{\frac{(xt)^2 + (yt)^2}{xt + yt}}$$

$$= \frac{t}{t^{1/2}}\sqrt{\frac{x^2+y^2}{x+y}}$$

$$\sin u = t^{1/2}\sqrt{\frac{x^2+y^2}{x+y}}$$

It is homogenus equation of degree $\frac{1}{2}$

By Euler's theorem

(1) $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}$$

$\therefore f(u) = \sin u, \quad n = \frac{1}{2}$

$f'(u) = \cos u$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{(\sin u)}{(\cos u)} = \frac{\tan u}{2}.$$

(2) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

$\therefore g(u) = \frac{\tan u}{2}$

$g'(u) = \frac{\sec^2 u}{2}$

$$= \frac{\tan u}{2}\left[\frac{\sec^2 u}{2} - 1\right]$$

$$= \frac{\tan u}{2}\left[\frac{1 + \tan^2 u}{2} - 1\right]$$

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$$= \frac{\tan y}{2}\left[\frac{1+\tan^2 y-2}{2}\right]$$

$$= \boxed{\frac{1}{4}\tan y\,(\tan^2 y-1)}$$